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On the giant magnon and spike solutions for strings on $AdS_3 \times S^3$

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Abstract: We study solutions for the rotating strings on the sphere with a background NS-NS field and on the anti-de-Sitter spacetime. We show the existence of magnon and single spike solutions on R$\times$S^2 in the presence of constant magnetic field as two limiting cases. We also study the solution for strings on AdS$_3\times$ S^3 with Melvin deformation. The dispersion relations among various conserved charges are shown to receive finite corrections due to the deformation parameter. We further study the rotating string on AdS$_3\times$ S^3 geometry with two conserved angular momenta on S^3 and one spin along the AdS$_3$. We show that there exists two kinds of solutions: a circular string solution and a helical string. We find out the dispersion relation among various charges and give physical interpretation of these solutions.

Keywords: Long strings, AdS-CFT Correspondence.

Contents

1. Introduction

A remarkable development in the field of string theory is the celebrated string theory-gauge theory duality, which relates the spectrum of free string on $AdS_5 \times S^5$ with that of operator dimension of $\mathcal{N} = 4$ super Yang-Mills (SYM) in planar limit. Determining this spectrum is an interesting and challenging problem. Recently it has been realized that this problem of counting the operators in gauge theory has an elegant formulation in terms of *integrable* spin chain [1 − 7]. In the dual formulation, the string theory has also integrable structure in the semiclassical limit. Recently Hofman and Maldacena (HM) considered a special limit where the problem of determining the spectrum on both sides simplifies considerably [8]. In this limit the 't Hooft coupling λ is held fixed allowing for a direct interpolation between the gauge theory ($\lambda \ll 1$) and string theory ($\lambda \gg 1$) and the energy E (or conformal dimension $\Delta = E$) and a U(1) R-charge J both become infinite with the difference $E - J$ held fixed. The spectrum consists of elementary excitations known as magnons that propagates with a conserved momentum p along the long spin chain [8].[1] These magnon excitations satisfy a dispersion relation of the type (in the large 't Hooft limit (λ))

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left|\sin\frac{p}{2}\right| \ . \tag{1.1}$$

A more general type of solution are the ones rotating in AdS$_5$, one of which is the spiky string [19] which describes the higher twist operators from field theory view point. Giant magnon solutions can be seen as a special limit of such spiky strings with shorter wavelength. Several papers [20 − 34] have been devoted in studying the gauge theory and string theory side of this interesting rotating solutions in AdS space and on the sphere. Hence

[1] for more work on related topic see for example [9 − 18]

it is very important to find out more general class of rotating and pulsating strings in $AdS_3\times S^3$ background and look for possible dual operators in the gauge theory. Because the complete understanding of the gauge theory operators corresponding to the semi classical string states is still lacking, it seems reasonable to find out the string states first and then look for possible operators in the dual side.

In this paper we study few examples of spike solutions for strings in $AdS_3\times S^3$ background in an attempt to study the string spectrum and the other elementary string states further. The $AdS_3\times S^3$ spacetime has been studied by using the $SL(2,R)\times SU(2)$ Wess-Zumino-Witten model. In the study of D-branes on this group manifold, one needs the mechanism of 'flux stabilisation' which ensures the stability of these branes against the collapse of the sphere. This has opened up a new window for the understanding of strings and branes in AdS space in the past. Our interest is whether one could find out any rotating string solution that looks like a spike and/or magnon in this background. As we will show in what follows that there exist such a solution, which modifies the relation between various conserved charges of the spike solution in a very natural way. Our next example is the spike solution in a Melvin deformed $AdS_3\times S^3$ background, and we will show that in the limit of small *deformation* parameter, to the leading order, the *energy Vs height* of spike, relationship gets corrections even at the lowest order in λ. Finally we present an interesting example of elementary string solution with one spin along AdS_3 and two angular momenta along S^3. We find a parameter space of configurations which admit two interesting classes of solution. One of them is a classical circular string on AdS_3 with the infinite spin S and at the same time, the giant magnon on S^3 with the finite angular momentum. The other is a helical string which has the same configuration with the circular string on the sphere but becomes an array of the spikes on AdS_3. We will show that these two solutions satisfy similar dispersion relation with two parameters, the velocity of the string v and the winding number $\tilde{w}$. In the absence of an exact expression for energy, we will write a perturbative expansion form of the dispersion relation and give the physical meaning of this solution. For the helical string case, we will find the dispersion relation for a single spike which is one segment of the helical string.

The rest of the paper is organized as follows. In section 2, we calculate the energy and momentum of a spike on a two sphere with a constant background NS-NS $B_{\mu\nu}$ field. We have shown that for the rigidly rotating string of the two sphere, in the background of constant NS-NS B-field, there exists two limiting cases of interest. First one is the known magnon solution studied in [35, 36]. The second one is the single spike solution that generalizes of the single spike solution on $R\times S^2$ found in [20]. We compute its energy E and angular momentum J and a function of $\bar{\theta}$ (the height of the spike) and the constant background field. Section 3 is devoted to the study of spike like solution in the magnetic Melvin deformed $AdS_3\times S^3$, where we constrain the motion of the string along $R\times S^3$ only. In the small deformation parameter, we show that the relationship between the angular momenta and the height of the spike, which is a generalization of the result obtained in [20]. In section 4 we calculate the example of multi spin spike solution in the $AdS_3\times$ S^3 background with two angular momenta along S^3 and a spin along AdS_3. We find two classes of solution of particular interest and the dispersion relation for each. These multi-

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spin solutions can be reinterpreted as a generalization of the giant magnon on S^2 with other spins and have different shapes on the AdS space. Finally in section 5, we conclude with some remarks.

2. Spike on $\mathrm{R}\times\mathrm{S}^2$ with a background NS-NS B field

As a first example we will show the existence of a single spike solution of the string around $\mathrm{R}\times\mathrm{S}^2$ with a background NS-NS B field. We will show that for the string rotating around the rigid sphere in the background of B field, there exists two interesting solution. The first one is a magnon solution found in [35], and the second one is a single spike solution which generalizes the results of [20] can be obtained from two different limits of the same solution. As explained in [37], the NS-NS background field has been used for the purpose of stability of the size of the sphere against its shrinking to zero size. The metric and the background NS-NS flux field is given by

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta d\phi^2, \quad B_{\theta\phi} = B\sin\theta \tag{2.1}$$

We are interested in finding out the classical rotating string solution around this geometry. To do so, as usual the starting point is to write down the Polyakov form of the action

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int_{-\pi}^{\pi} d\sigma d\tau \left[\sqrt{-\gamma}\gamma^{\alpha\beta} g_{MN}\partial_\alpha x^M \partial_\beta x^N - e^{\alpha\beta}\partial_\alpha x^M \partial_\beta x^N B_{MN}\right] \tag{2.2}$$

and where $\gamma^{\alpha\beta}$ is world-sheet metric and $e^{\alpha\beta}$ is the anti symmetric tensor defined as $e^{01} = -e^{10} = 1$. Finally, the modes $x^M, M = 1,\ldots,9$ parameterize the embedding of the string in the background. The equations of motion derived by the above action has to be supplemented by the following Virasoro constraints

$$\begin{aligned} g_{MN}\left(\partial_\sigma x^M \partial_\sigma x^N + \partial_\tau x^M \partial_\tau x^N\right) &= 0 \\ g_{MN}\partial_\sigma x^M \partial_\tau x^N &= 0 \end{aligned} \tag{2.3}$$

We consider a spike string in the following worldsheet parametrization

$$t = \kappa\tau, \quad \theta = \theta(y), \quad \phi = \omega\tau + \tilde{\phi}(y). \tag{2.4}$$

where $y = \alpha\sigma + \beta\tau$. With this the Virasoro constraints take the form

$$\begin{aligned} \dot{\theta}\theta' + \sin^2\theta\dot{\phi}\phi' &= 0 \\ -(\dot{t}^2 + t'^2) + \dot{\theta}^2 + \theta'^2 + \sin^2\theta(\dot{\phi}^2 + \phi'^2) &= 0 \end{aligned} \tag{2.5}$$

The next step is to use the above ansatz in the equations of motion[2] and using Virasoro constraints one can obtain

$$\tilde{\phi}' = \frac{1}{(\alpha^2-\beta^2)}\left(\beta\omega - \frac{\beta\kappa^2}{\omega\sin^2\theta}\right) \tag{2.6}$$

$$\theta'^2 = \frac{\sin^2\theta}{(\alpha^2-\beta^2)^2}\left(\alpha^2 - \frac{\beta^2\kappa^2}{\omega^2\sin^2\theta}\right)\left(\frac{\kappa^2}{\sin^2\theta} - \omega^2\right). \tag{2.7}$$

[2] one can check that with the choice of the B- field in eq. (2.1), its contribution to the equations of motion cancel among each other.

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In order to find spike like solutions, let us define

$$\sin\theta_0 = \frac{\beta\kappa}{\alpha\omega}, \qquad \sin\theta_1 = \frac{\kappa}{\omega}. \tag{2.8}$$

So now using these definitions one can rewrite the above equations as follows

$$\tilde{\phi}' = \frac{\beta\omega}{(\alpha^2-\beta^2)\sin^2\theta}\left(\sin^2\theta - \sin^2\theta_1\right) \tag{2.9}$$

and

$$\theta' = \frac{\omega\alpha}{(\alpha^2-\beta^2)\sin\theta}\sqrt{(\sin^2\theta_0 - \sin^2\theta)(\sin^2\theta - \sin^2\theta_1)}. \tag{2.10}$$

The two conserved quantities, namely the total energy and angular momentum are defined as

$$\begin{aligned} E &= 2T\frac{\kappa}{\alpha}\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'} && (2.11)\\ J &= 2\frac{T}{\alpha}\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}(\sin^2\theta\dot{\phi} + B_{\theta\phi}\theta'). && (2.12)\end{aligned}$$

Now we will consider two limits will define the giant magnon and the single spike around this R×S^2.
1. For giant magnon we put $\sin^2\theta_1 = 1$, which implies that

$$\begin{aligned} E - J &= 2T(1+B)\cos\theta_0,\\ \Delta\phi &= \int_{\theta_0}^{\pi/2}\frac{d\theta}{\theta'}\tilde{\phi}' = 2\,\mathrm{arcsin}(\cos\theta_0) = \pi - 2\theta_0, \end{aligned} \tag{2.13}$$

so the giant magnon dispersion relation as mentioned in [35] can be evaluated as

$$E - J = 2T(1+B)\cos\theta_0 = \frac{\sqrt{\lambda}}{\pi}(1+B)\sin\frac{\Delta\phi}{2}. \tag{2.14}$$

Note that the dispersion relation gets modified due to presence of B field [35].
2. For the single spike solution, we consider the opposite limit $\sin^2\theta_0 = 1$. This implies that

$$J = 2\frac{T}{\alpha}\int_{\pi/2}^{\theta_1}\frac{d\theta}{\theta'}(\sin^2\theta\dot{\phi} + B_{\theta\phi}\theta'). \tag{2.15}$$

One can evaluate the above integral and obtain

$$J = 2T(1+B)\cos\theta_1 = \frac{\sqrt{\lambda}}{\pi}(1+B)\cos\theta_1. \tag{2.16}$$

Hence one can show that

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi}\left(\frac{\pi}{2} - \theta_1\right). \tag{2.17}$$

Now the height of the spike can be defined as

$$\bar{\theta} = \left(\frac{\pi}{2} - \theta_1\right) \tag{2.18}$$

As usual the energy of the spike can be defined as

$$\Delta = (E - T\Delta\phi) - J = \frac{\sqrt{\lambda}}{\pi}(\bar{\theta} - (1+B)\sin\bar{\theta}). \tag{2.19}$$

Notice that this relationship also gets a correction due to the presence of the background B field. Putting B = 0, we get the result derived in [20]. The generalization of this solution by adding one more angular momentum to get a solution on $\mathrm{R}\times\mathrm{S}^3$ is straightforward. We however leave this as an exercise for the readers.

3. Rotating string on the Melvin deformed $\mathbf{AdS_3\times S^3}$

Recently, the rotating string with spin along various directions of S^5 was investigated by many authors in the $\mathrm{AdS}_5\times\mathrm{S}^5$ background [20, 36, 40, 25]. As mentioned earlier, the rotating string appears as a magnon solution which is a smooth configuration or a spike solution with cusp. Here, we will consider a string rotating on S^3 of the Melvin field deformed $\mathrm{AdS}_3\times\mathrm{S}^3$ background (see [20] for the rigidly rotating string on S^3 with no deformation).

The relevant metric on $\mathrm{R}\times\mathrm{S}^3$ with such a deformation is given by [36]

$$ds^2 = \sqrt{1+B^2\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta d\phi^2 + \frac{\cos^2\theta}{1+B^2\cos^2\theta}d\chi^2\right) \tag{3.1}$$

On this background, the string is rotating in two direction, ϕ and χ, is described by the Nambu-Goto action

$$S = T\int d^2\sigma\mathcal{L} = T\int d^2\sigma\sqrt{(\partial_\sigma X\cdot\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2}. \tag{3.2}$$

The equations of motion of this system are

$$\begin{aligned}
\partial_\sigma\frac{\partial\mathcal{L}}{\partial t'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{t}} &= \frac{\partial\mathcal{L}}{\partial t}\\
\partial_\sigma\frac{\partial\mathcal{L}}{\partial\theta'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{\partial\mathcal{L}}{\partial\theta}\\
\partial_\sigma\frac{\partial\mathcal{L}}{\partial\phi'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \frac{\partial\mathcal{L}}{\partial\phi}\\
\partial_\sigma\frac{\partial\mathcal{L}}{\partial\chi'} + \partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{\chi}} &= \frac{\partial\mathcal{L}}{\partial\chi},
\end{aligned} \tag{3.3}$$

where $\cdot$ or $\prime$ means the derivative with respect to τ or σ, respectively. We choose the following parametrization,

$$t = \kappa\tau, \quad \theta = \theta(\sigma), \quad \phi = \omega_1\tau + \sigma, \quad \chi = \chi(\sigma) + \omega_2\tau \tag{3.4}$$

the first and the third equations of motion reduce the following forms

$$\frac{\partial\mathcal{L}}{\partial t'} = c_1, \quad \frac{\partial\mathcal{L}}{\partial\phi'} = c_2, \tag{3.5}$$

where c_1 and c_2 are arbitrary constants.

From these equations with two integration constants, we can obtain

$$\chi'(\sigma) = \frac{\left\{\kappa(c_1\kappa - c2\omega_1) + (B^2\kappa(c_1\kappa - c_2\omega_1) - c_1\omega_2^2)\cos^2\theta\right\}\tan^2\theta}{\omega_1^2\sin^2\theta - \kappa^2}. \tag{3.6}$$

For $\theta = \pi/2$, χ' becomes singular so that we choose the integration constants to cancel this singularity. If two constants satisfy $(c_1\kappa - c_2\omega_1) = 0$, then χ' is not singular any more. From now on, we choose $c_1 = \omega_1$ and $c_2 = \kappa$ for simplicity. Using these fixed integration constants, the equations for $\chi'(\sigma)$ and $\theta'(\sigma)$ reduced to[3]

$$\begin{aligned}
\chi'(\sigma) &= \frac{\omega_1\omega_2\sin^2\theta}{\omega_1^2\sin^2\theta - \kappa^2}, \\
\theta'(\sigma) &= \frac{\kappa\sin\theta\cos\theta\sqrt{(\omega_1^2 - \omega_2^2)\sin^2\theta - \kappa^2 - B^2\sin^2\theta(\omega_1^2\sin^2\theta - \kappa^2)}}{\omega_1^2\sin^2\theta - \kappa^2}.
\end{aligned} \tag{3.7}$$

At the fixed time, the string configuration is determined from the above equations.

From now on, we consider the string configuration in the (ϕ,θ) space in the small B limit. Note that the second equation in eq. (3.7) is meaningful only when the inside of the square root becomes a non-negative value, which gives a constraint to the range of θ.

The exact positive values of $\sin\theta$ making the square root zero are

$$\sin\theta = \sqrt{\frac{\omega_1^2 - \omega_2^2 + B^2\kappa^2 \pm \sqrt{(\omega_1^2 - \omega_2^2 + B^2\kappa^2)^2 - 4\kappa^2 B^2\omega_1^2}}{2B^2\omega_1^2}}. \tag{3.8}$$

Assuming that $B^2 \ll \omega_1^2 - \omega_2^2$ and $\omega_1^2 - \omega_2^2 > \kappa^2$, then the range of $\sin\theta$ making the inside of the square root a non-negative value is given by $\sin\theta_{\rm min} \leq \sin\theta \leq \sin\theta_{\rm max}$ where

$$\begin{aligned}
\sin\theta_{\rm min} &= \frac{\kappa}{\sqrt{\omega_1^2 - \omega_2^2}}\left(1 + \frac{\kappa^2\omega_2^2 B^2}{2(\omega_1^2 - \omega_2^2)^2}\right) + \mathcal{O}(B^4) \\
&\equiv \sin\theta_0 + \frac{\kappa^3\omega_2^2 B^2}{2(\omega_1^2 - \omega_2^2)^{5/2}} + \mathcal{O}(B^4), \\
\sin\theta_{\rm max} &= \frac{\sqrt{\omega_1^2 - \omega_2^2}}{B\omega_1}\left(1 - \frac{\kappa^2\omega_2^2 B^2}{2(\omega_1^2 - \omega_2^2)^2} + \mathcal{O}(B^4)\right),
\end{aligned} \tag{3.9}$$

where we set $\sin\theta_0 = \frac{\kappa}{\sqrt{\omega_1^2 - \omega_2^2}}$ which is the minimum value in the case $B = 0$. Due to the above assumption, $\sin\theta_{\rm max}$ is always greater than 1, so the relevant range of $\sin\theta$ becomes $\sin\theta_{\rm min} \leq \sin\theta \leq 1$. In eq. (3.7), θ' has a singularity at $\sin\theta = \sqrt{\kappa/\omega_1}$ which corresponds to the peak of the spike solution. Since $\sqrt{\kappa/\omega_1} < \sin\theta_{\rm min}$, this singular point is always located at the outside of the relevant range of θ, which implies that there is no spike solution in the assumed parameter region.[4]

[3]The second order differential equations for χ and for θ are very complicated indeed. Hence we first solve the first and third equation in (3.3) and then use that to write the first order equations for χ and θ. We have checked that they all are consistent with each other. A similar analysis was presented in [20].

[4]see a comment on the similarity between the giant magnon and the spike solution with two angular momenta in [20]

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Note that since $\phi' = 1$ from the eq. (3.4), θ' is equivalent to $\frac{\partial\theta}{\partial\phi}$. At two boundary values of θ, $\theta_{\rm min}$ and $\pi/2$, θ' is zero, so we can identify these two points with the top and the bottom of the giant magnon. In addition, in the (χ, θ) space we can also find a similar string configuration using the $\frac{\partial\theta}{\partial\chi}$ equation

$$\frac{\partial\theta}{\partial\chi} = \frac{\kappa\cos\theta\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2-B^2\sin^2\theta(\omega_1^2\sin^2\theta-\kappa^2)}}{\omega_1\omega_2\sin\theta}. \tag{3.10}$$

As a result, the macroscopic string found here is a giant magnon in both ϕ and χ directions.

The energy of this giant magnon is given by

$$E = 2T\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\frac{\partial\mathcal{L}}{\partial\dot{t}} = 2T\int_{\theta_0}^{\theta_1}d\theta\frac{(\omega_1^2-\kappa^2-B^2\kappa^2\cos^2\theta)\tan\theta}{\kappa\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2-B^2\sin^2\theta(\omega_1^2\sin^2\theta-\kappa^2)}}, \tag{3.11}$$

where $\theta_1 = \pi/2$ and $\theta_0 = \theta_{\rm min}$ and the first angular momentum in the ϕ direction is

$$J_1 = 2T\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 2T\int_{\theta_0}^{\theta_1}d\theta\frac{\omega_1\cos\theta\sin\theta(1-B^2\sin^2\theta)}{\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2-B^2\sin^2\theta(\omega_1^2\sin^2\theta-\kappa^2)}}. \tag{3.12}$$

The last conserved quantity is the second angular momentum in the χ direction given by

$$J_2 = 2T\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\frac{\partial\mathcal{L}}{\partial\dot{\chi}} = 2T\int_{\theta_0}^{\theta_1}d\theta\frac{\omega_2\cos\theta\sin\theta}{\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2-B^2\sin^2\theta(\omega_1^2\sin^2\theta-\kappa^2)}}. \tag{3.13}$$

The difference in angle between two bottoms (or top) of the giant magnon, corresponding to the size of the giant magnon, becomes

$$\Delta\Theta = 2\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'} = 2\int_{\theta_0}^{\theta_1}d\theta\frac{\omega_1^2\sin^2\theta-\kappa^2}{\kappa\sin\theta\cos\theta\sqrt{(\omega_1^2-\omega_2^2)\sin^2\theta-\kappa^2-B^2\sin^2\theta(\omega_1^2\sin^2\theta-\kappa^2)}}. \tag{3.14}$$

Using this, the integration of $E - T\Delta\Theta$ in this small B limit becomes

$$E - T\Delta\Theta \approx 2T\left\{\bar{\theta} - \frac{\bar{\kappa}\sin\gamma B}{2\cos^2\gamma}\right\}, \tag{3.15}$$

where $\bar{\theta} = \frac{\pi}{2} - \theta_0$, $\bar{\kappa} = \frac{\kappa}{\omega_1}$ and $\sin\gamma = \frac{\omega_2}{\omega_1}$. Two angular momentums, J_1 and J_2 are given by

$$\begin{aligned} J_1 &\approx 2T\left\{\frac{1}{\cos\gamma}\sin\bar{\theta} - \frac{\bar{\kappa}^2\sin\gamma B}{2\cos^4\gamma}\right\}, \\ J_2 &\approx 2T\left\{\frac{\sin\gamma}{\cos\gamma}\sin\bar{\theta} - \frac{\bar{\kappa}^2\sin^2\gamma B}{2\cos^4\gamma}\right\}. \end{aligned} \tag{3.16}$$

When $B = 0$, all conserved quantities are reduced to those on the undeformed S^3 [20].

$$J_1^2 \approx J_2^2 + \frac{\lambda}{\pi^2}\sin^2\bar{\theta} - \frac{\lambda}{\pi^2}\frac{\bar{\kappa}^2\sin\gamma\sin\bar{\theta}}{\cos^3\gamma}B \tag{3.17}$$

Again in B = 0 limit it reduces to the result obtained in [20] for the three sphere case.

4. Three-spin spiky string on $\mathrm{AdS}_3 \times \mathrm{S}^3$

Here, we consider a three-spin string solution in $\mathrm{AdS}_3\times \mathrm{S}^5$ which has one spin S in AdS_3 and two spins J_1 and J_2 in S^3. In [40], the three-spin giant magnon in the special parameter region was investigated in the same background. In this section, we will consider a different solution in the different parameter region which is not smoothly connected with the case in ref. [40].

Now, we consider the relevant metric of $\mathrm{AdS}_3\times \mathrm{S}^3$ as a subspace of $\mathrm{AdS}_5\times \mathrm{S}^5$

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2 + d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2 \ . \tag{4.1}$$

In the conformal gauge, the Polyakov string action is given by

$$\begin{aligned} I = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \Big[& -\cosh^2\rho \ \ (t'^2 - \dot{t}^2) + \rho'^2 - \dot{\rho}^2 + \sinh^2\rho \ \ (\phi'^2 - \dot{\phi}^2) \\ & + (\theta'^2 - \dot{\theta}^2) + \cos^2\theta \ \ ({\phi_1'}^2 - {\dot{\phi}_1}^2) + \sin^2\theta \ \ ({\phi_2'}^2 - {\dot{\phi}_2}^2)\Big] , \end{aligned} \tag{4.2}$$

where dot and prime denote the derivatives with respect to τ and σ respectively. Now, we choose the following parametrization for a rotating string in the above background

$$\begin{aligned} &t = \tau + h_1(y), && \rho = \rho(y), && \phi = w(\tau + h_2(y)), \\ &\phi_1 = \tau + g_1(y), && \theta = \theta(y), && \phi_2 = \tilde{w}(\tau + g_2(y)), \end{aligned} \tag{4.3}$$

where $y = \sigma - v\tau$.

After introducing the appropriate integration constants, the equations of motion for the S^3 part are reduced to

$$\begin{aligned} \partial_y g_1 &= \frac{v}{1-v^2}\tan^2\theta, \\ \partial_y g_2 &= -\frac{v}{1-v^2}, \\ \partial_y\theta &= \frac{\sin\theta}{(1-v^2)\cos\theta}\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}. \end{aligned} \tag{4.4}$$

For the consistency, θ should run from 0 to $\theta_{\max} = \arccos\frac{v}{\sqrt{1-\tilde{w}^2}}$, when $1-\tilde{w}^2 > v^2$. The solution of the last equation is given by

$$\sin\theta = \frac{\alpha}{\cosh\beta y}, \tag{4.5}$$

where $\alpha = \sqrt{\frac{1-v^2-\tilde{w}^2}{1-\tilde{w}^2}}$ and $\beta = \frac{\sqrt{1-v^2-\tilde{w}^2}}{1-v^2}$ [40]. Note that since at $\tau = 0$, $\theta = 0$ corresponds to $\sigma = \pm\infty$ and $\theta_{\max}$ is described by $\sigma = 0$, so the range of σ is given by $-\infty < \sigma < \infty$.

The string configuration in θ and ϕ_1 space is described by

$$\frac{\partial\theta}{\partial\phi_1} = \frac{\cos\theta}{v\sin\theta}\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}. \tag{4.6}$$

Note that the equator of S^3 is located at $\theta = 0$ and at this equator the above equation is singular. When $\theta = 0$ or $\theta = \theta_{\max}$, $\frac{\partial\theta}{\partial\phi_1}$ becomes ∞ or 0 respectively, which implies that the string shape of this solution described by θ and ϕ_1 looks like that of giant magnon on S^2. The angle difference of this S^2 magnon-like solution in the ϕ_1 direction reads

$$\Delta\phi_1 = 2\int_0^{\theta_{\max}} d\theta \frac{v\sin\theta}{\cos\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}} = 2\theta_{\max}. \tag{4.7}$$

Now, we consider the open string configuration in the AdS_3 part. After some calculation, the equations for t and ϕ becomes

$$\begin{aligned} \partial_y h_1 &= \frac{1}{1-v^2}\left(-v + \frac{d_1}{\cosh^2\rho}\right), \\ \partial_y h_2 &= \frac{1}{1-v^2}\left(-v + \frac{d_2}{\sinh^2\rho}\right), \end{aligned} \tag{4.8}$$

where d_1 and d_2 are integration constants. The case having $d_2 = 0$ has been studied in [40]. Here, we consider the different parameter region, $d_2 \neq 0$ which gives the different type of the string solution. Using the relation $d_1 = v + w^2 d_2$, the Virasoro constraints are reduced to a single equation

$$(\partial_y\rho)^2 = \frac{w^2}{v}(1 - v\partial_y h_2)\partial_y h_2 \sinh^2\rho - \frac{1}{v}(1 - v\partial_y h_1)\partial_y h_1 \cosh^2\rho. \tag{4.9}$$

Notice that the variation of this Virasoro constraint equation with respect to y gives the equation of motion for ρ

$$\partial_y^2\rho = -\frac{\sinh\rho\cosh\rho}{(1-v^2)^2}\left[w^2\left(1 - \frac{d_2^2}{\sinh^4\rho}\right) - \left(1 - \frac{d_1^2}{\cosh^4\rho}\right)\right]. \tag{4.10}$$

Hence, to obtain $\partial_y\rho$ it is sufficient to solve the above Virasoro constraint instead of the equation of motion for ρ. The general form of $\partial_y\rho$ is given by

$$\partial_y\rho = \pm\frac{A}{(1-v^2)\cosh\rho\sinh\rho}, \tag{4.11}$$

where

$$A = \sqrt{(1-w^2)\sinh^6\rho + (1-v^2-w^2)\sinh^4\rho + d_2 w^2(2v - d_2(1-w^2))\sinh^2\rho - d_2^2 w^2}. \tag{4.12}$$

Actually, it is difficult to calculate the physical quantities like the energy and the angular momentum using the above A, so we choose a special set of parameters like $w^2 = 1 - v^2$ and $d_2 = \frac{2v}{1-w^2} = \frac{2}{v}$, which removes the second and the third term in A. Then, the eq. (4.12) reduces to a simple form

$$A = \sqrt{(1-w^2)\sinh^6\rho - d_2^2 w^2}, \tag{4.13}$$

which gives the minimum value of $\rho_{\min} = \text{arcsinh}\left(\frac{d_2^2 w^2}{1-w^2}\right)^{1/6} = \text{arcsinh}\left(\frac{4(1-v^2)}{v^4}\right)^{1/6}$ for $1 - w^2 > 0$. Since $\rho_{\min}$ goes to zero (infinity) as $v \to 1(0)$ respectively, we will consider the range of v as $0 < v < 1$.

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Using this reduced function A and eq. (4.8), the following differential equation

$$\frac{\partial\rho}{\partial\phi} = \frac{A\ \sinh\rho}{\cosh\rho(d_2 - v\sinh^2\rho)}, \tag{4.14}$$

describes the shape of the string on the AdS part. As will be shown in the next sections, this gives two kinds of the string solution: one is the circular string rotating at $\rho_{\rm min}$ and the other is the helical string extended from $\rho_{\rm min}$ to $\rho_{\rm max} = \infty$ with the infinite winding number and the infinite angular momentum S in the ϕ-direction.

4.1 Circular string on AdS

The simple solution of eq. (4.14) is given by the string located at $\rho = \rho_{\rm min}$ where A is zero. From eq. (4.8) at a fixed time $\tau = 0$ where $y = \sigma$, the string configuration in ϕ-direction is given by

$$\phi = \frac{1}{1-v^2}\left(\left(\frac{2v}{1-v^2}\right)^{1/3} - v\right)\sigma. \tag{4.15}$$

Note that the coefficient of this relation is not zero except $v = 0$. Since the range of σ is $-\infty < \sigma < \infty$, ϕ also has to cover the range, $-\infty < \phi < \infty$. This implies that this solution describes the circular string having the infinite windings. The conserved charges of this string are given by

$$\begin{aligned}
E &= \frac{\sqrt{\lambda}}{2\pi}\int dy \frac{\cosh^2\rho_{\rm min} - d_1 v}{1-v^2} = \frac{\sqrt{\lambda}}{\pi}\int_0^{\theta_{\rm max}} d\theta \frac{\cos\theta(\cosh^2\rho_{\rm min} - 2 + v^2)}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}, \\
S &= \frac{w\sqrt{\lambda}}{2\pi}\int dy \frac{\sinh^2\rho_{\rm min} - d_2 v}{(1-v^2)} = \frac{w\sqrt{\lambda}}{\pi}\int_0^{\theta_{\rm max}} d\theta \frac{\cos\theta(\sinh^2\rho_{\rm min} - 2)}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}, \\
J_1 &= \frac{\sqrt{\lambda}}{2\pi}\int dy \frac{\cos^2\theta - v^2}{(1-v^2)} = \frac{\sqrt{\lambda}}{\pi}\int_0^{\theta_{\rm max}} d\theta \frac{\cos\theta(\cos^2\theta - v^2)}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}, \\
J_2 &= \frac{\tilde{w}\sqrt{\lambda}}{2\pi}\int dy\ \frac{\sin^2\theta}{(1-v^2)} = \frac{\tilde{w}\sqrt{\lambda}}{\pi}\int_0^{\theta_{\rm max}} d\theta \frac{\cos\theta\sin\theta}{\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}.
\end{aligned} \tag{4.16}$$

In the above integral equations, $\sin\theta$ in the denominator gives rise to the logarithmic divergence at $\theta = 0$, so three charges, E, S and J_1 have a logarithmic divergence where as J_2 is finite. Interestingly, these quantities satisfy the following relation

$$E - \frac{S}{w} = \frac{1+v^2}{1-v^2}\left(J_1 + \frac{J_2}{\tilde{w}}\right), \tag{4.17}$$

which is the exact dispersion relation of the string on $AdS_3\times S^3$ with two parameters, v and $\tilde{w}$ and the finite charge, J_2 is given by

$$J_2 = \frac{\sqrt{\lambda}}{\pi}\frac{\tilde{w}}{\sqrt{1-\tilde{w}^2}}\sin\theta_{\rm max}. \tag{4.18}$$

To investigate this solution more clearly, we consider the special parameter limit $v = 0$ and $\tilde{w} = 0$, where $\rho_{\rm min} \to \infty$ and $\theta_{\rm max} = \frac{\pi}{2}$. Here, $\tilde{w} = 0$ implies that the string solution has to a point-like configuration in the ϕ_2-direction because the angular momentum J_2 and the angle difference $\Delta\phi_2$ vanishes. Hence, in this parameter region, the string solution reduces to that on $\mathrm{AdS}_3 \times S^2$. The shape of this solution on S^2 is described by the relation between θ and σ

$$\tan\frac{\theta}{2} = \mathrm{e}^{\sigma}, \tag{4.19}$$

which is obtained by calculating the integral of the last equation in eq. (4.4) at $\tau = 0$. Since $\theta_{\max} = \pi/2$, the angle difference $\Delta\phi_1$ becomes π from eq. (4.7), which gives the shape of a giant magnon on S^2 with the maximal ϕ_1-angle difference π. As a result, this solution describes a circular string rotating at $\rho_{\min}$ with the infinite angular momentum S and having the shape of the magnon on S^2, whose dispersion relation becomes

$$E - S - J_1 = \frac{\sqrt{\lambda}}{\pi}. \tag{4.20}$$

For the giant magnon on S^2 [19, 8, 41] with the following dispersion relation

$$E - J_1 = \frac{\sqrt{\lambda}}{\pi}, \tag{4.21}$$

this string has no angular momentum in the ϕ-direction and is located at $\rho = 0$. So the circular string in the limit $v = 0$ and $\tilde{w} = 0$, can be considered as an extension of the giant magnon on S^2 extended in the ϕ-direction with the infinite winding number and the infinite angular momentum.

To describe the string solution on $\mathrm{AdS}_3\times \mathrm{S}^3$, we should turn on the angular momentum J_2, which corresponds to considering the parameter region with $\tilde{w} \neq 0$. In the case of $\tilde{w} \neq 0$ and $v = 0$, the dispersion relation becomes

$$E - S - J_1 = \left.\frac{J_2}{\tilde{w}}\right|_{v=0} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\sqrt{1-\tilde{w}^2}}. \tag{4.22}$$

For $v \neq 0$, the above dispersion relation can have some corrections ΔE

$$E - S - J_1 = \frac{J_2}{\tilde{w}} + \Delta E, \tag{4.23}$$

where

$$\Delta E = \frac{1-\sqrt{1-v^2}}{\sqrt{1-v^2}}S + \frac{2v^2}{1-v^2}J, \tag{4.24}$$

and we set $J = J_1 + J_2/\tilde{w}$.

Note that all conserved charges defined in eq. (4.16) are functions of λ, v and $\tilde{w}$. Since the dependence of λ can be removed by a simple rescaling, we can consider these charges as functions of v and $\tilde{w}$ effectively. This implies that in principle two parameters, v and $\tilde{w}$ can be rewritten as functions of S and $J = J_1 + J_2/\tilde{w}$. In the limit $v \to 0$ where $\rho_{\min} \to \infty$, S and J can be approximately rewritten as

$$\begin{aligned} S &\approx \left(\frac{2^{2/3}}{v^{4/3}} + \mathcal{O}(v^{2/3})\right)\Delta, \\ J &\approx \left(1 + \mathcal{O}(v^2)\right)\Delta, \end{aligned} \tag{4.25}$$

where

$$\Delta = \frac{\sqrt{\lambda}}{\pi} \int_0^{\theta_{\max}} d\theta \frac{\cos\theta}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}. \tag{4.26}$$

Then, we can approximately rewrite v in terms of S and J

$$v^2 \approx 2\left(\frac{J}{S}\right)^{3/2}. \tag{4.27}$$

So the dispersion relation becomes in this approximation

$$E = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\sqrt{1-\tilde{w}^2}}\sin\theta_{\max} + S\left[1+\left(\frac{J}{S}\right)^{3/2}\right] + J\left[1+4\left(\frac{J}{S}\right)^{3/2}\right] + \mathcal{O}\left(\frac{J}{S}\right)^3, \tag{4.28}$$

which describes the circular string rotating at $\rho_{\min}$ on the AdS and the magnon on S^2 with the finite angular momentum J_2.

4.2 Helical string on AdS

When the string is extended in the radial direction of the AdS space, ρ becomes a function of σ and $A \neq 0$. As shown in eq. (4.5), $0 < \theta < \pi$ covers the full range of σ, $-\infty < \sigma < \infty$ but unlike θ the range of ρ, $\rho_{\min} < \rho < \infty$ does not cover the full region of σ. In the asymptotic region, $\rho \to \infty$, the solution of eq. (4.11) at $\tau = 0$ is given by

$$\sigma - \sigma_n \sim \mathrm{e}^{-\rho}, \tag{4.29}$$

where σ_n is an integration constant. This implies that when $\rho \to \infty$ σ should go to σ_n, so $\rho_{\min} < \rho < \infty$ covers the finite range of σ only. As will be shown, this finite range of σ corresponds to that of one AdS spike solution in which $\rho_{\min}$ and $\rho_{\max}$ corresponds to the bottom of the valley between spikes and the cusp of the spike, respectively, See figure 1. To cover all σ, we should include the infinitely many spikes, the helical string means an array of infinite spikes on the AdS part.

The charges of the helical string are given by

$$\begin{aligned}
E &= \frac{\sqrt{\lambda}}{\pi}\int_0^{\theta_{\max}} d\theta \frac{\cos\theta(\cosh^2\rho - 2 + v^2)}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}, \\
S &= \frac{w\sqrt{\lambda}}{\pi}\int_0^{\theta_{\max}} d\theta \frac{\cos\theta(\sinh^2\rho - 2)}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}, \\
J_1 &= \frac{\sqrt{\lambda}}{\pi}\int_0^{\theta_{\max}} d\theta \frac{\cos\theta(\cos^2\theta - v^2)}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}, \\
J_2 &= \frac{\tilde{w}\sqrt{\lambda}}{\pi}\int_0^{\theta_{\max}} d\theta \frac{\cos\theta\sin\theta}{\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}},
\end{aligned} \tag{4.30}$$

where ρ is a complicate function of θ given by

$$\frac{d\rho}{d\theta} = \frac{A}{\cosh\rho\sinh\rho}\frac{\cos\theta}{\sin\theta\sqrt{(1-\tilde{w}^2)\cos^2\theta - v^2}}. \tag{4.31}$$

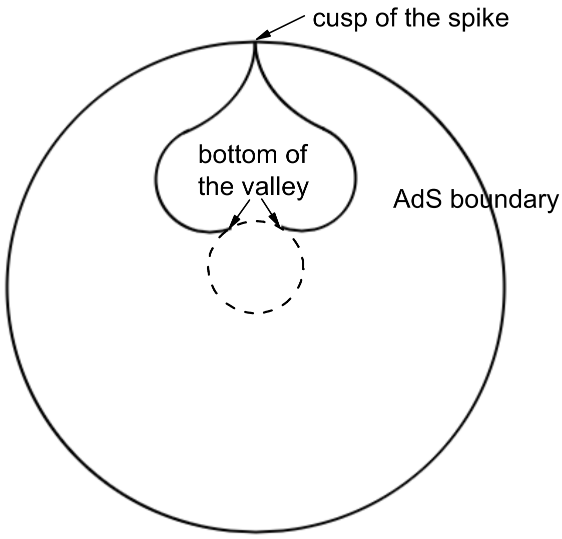

Figure 1: One spike of the helical string where the inner and outer circles indicate $\rho_{\min}$ and $\rho = \infty$, respectively.

Note that since θ covers all range of σ, these quantities in eq. (4.30) including the effect of the infinite AdS spikes corresponds to the charges of the helical string.

Before studying the dispersion relation of this helical string, we first concentrate on the one AdS spike solution which is just one segment of the helical string. As previously mentioned, the integral range of ρ covers only one spike, so it is useful to replace the integral measure $d\theta$ in eq. (4.30) with $d\rho$ for investigating the dispersion relation of the one spike. Using eq. (4.31), we can rewrite the energy and angular momentum of one AdS spike as

$$\begin{aligned} E^n &= \frac{\sqrt{\lambda}}{\pi} \int_{\rho_{\min}}^{\infty} d\rho \frac{\cosh\rho \sinh\rho \left(\cosh^2\rho - d_1 v\right)}{A}, \\ S^n &= \frac{\sqrt{\lambda}}{\pi} \int_{\rho_{\min}}^{\infty} d\rho \frac{w \cosh\rho \sinh\rho \left(\sinh^2\rho - d_2 v\right)}{A}, \end{aligned} \tag{4.32}$$

where the superscript n implies the n-th AdS spike. After performing the integral, the exact results becomes

$$E^n = \frac{\sqrt{\lambda}}{\pi v} \left(\sinh\rho_{\max} - \frac{\sqrt{\pi}\sinh\rho_{\min}\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})} - \frac{\sqrt{\pi}w^2\Gamma(\frac{7}{6})}{\sinh\rho_{\min}\,\Gamma(\frac{2}{3})} \right), \tag{4.33}$$

and

$$S^n = \frac{w\sqrt{\lambda}}{\pi v} \left(\sinh\rho_{\max} - \frac{\sqrt{\pi}\sinh\rho_{\min}\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})} - \frac{2\sqrt{\pi}\Gamma(\frac{7}{6})}{\sinh\rho_{\min}\,\Gamma(\frac{2}{3})} \right), \tag{4.34}$$

where Γ implies the gamma function and $\rho_{\max} = \infty$. So these two quantities diverge. The angle difference in ϕ-direction ($\Delta\phi \equiv -2\int_{\rho_{\min}}^{\infty} d\rho \frac{d\phi}{d\rho}$, where we insert a minus sign for the convenience) is given by

$$\Delta\phi = -2w \int_{\rho_{\min}}^{\infty} d\rho \frac{\cosh\rho\left(d_2 - v\sinh^2\rho\right)}{\sinh\rho\ A},$$
$$= 2w\left(\frac{\sqrt{\pi}\,\Gamma(\frac{7}{6})}{\sinh\rho_{\min}\,\Gamma(\frac{2}{3})} - \frac{\pi}{3v^2\sinh^3\rho_{\min}}\right). \tag{4.35}$$

The dispersion relation for one spike becomes

$$E^n - \frac{S^n}{w} = \frac{\sqrt{\lambda}}{\pi}\frac{1+v^2}{v\sqrt{1-v^2}}\left(\frac{\Delta\phi}{2} - \frac{\pi}{6}\right). \tag{4.36}$$

Interestingly, the charges of the helical string corresponding to the combination of the infinite AdS spikes with infinite S in ϕ-direction and the magnon on S^2 with the finite angular momentum J_2, also satisfies the same dispersion relation given in eq. (4.17). Using eq. (4.18), the dispersion relation for the full range of σ on $\mathrm{AdS}_3 \times S^3$ can be rewritten as

$$E - \frac{S}{w} - \frac{1+v^2}{1-v^2}J_1 = \frac{\sqrt{\lambda}}{\pi}\frac{1+v^2}{1-v^2}\frac{1}{\sqrt{1-\tilde{w}^2}}\sin\frac{p}{2}, \tag{4.37}$$

where we identify the angle difference $\Delta\phi_1(= 2\theta_{\max})$ corresponding to the size of the magnon on S^3 with the string world sheet momentum p. In the limit $v = 0$, since $\rho_{\min} \to \infty$, the size of one spike $\Delta\phi$ becomes zero, so the helical string configuration with infinitely many AdS spikes becomes a circular string studied in the previous section.

5. Discussion

We have studied, in this paper, new spike like solutions for strings moving on a sphere in a magnetic field background and on $\mathrm{AdS}_3\times \mathrm{S}^3$ geometry. First we have studied the solutions for rigid string moving on a S^2 with a constant background magnetic field. They can be classified as slowly moving strings, which are potentially different from the fast rotating strings. We have shown that this admits two limiting solutions of interest, the already studied magnons and the single spike, which infinitely wrap around the equator. The energy of the single spike solution has been shown to be modified due to the background field. The second example is the rotating solution for string moving in the Melvin deformed $\mathrm{AdS}_3\times \mathrm{S}^3$. In this case it is rather difficult to obtain the exact expression for the energy of the giant magnon. So we have taken the series expansion for charges in the small deformation parameter and then have found the perturbative expression to the leading order in the deformation parameter $\mathcal{O}(B)$.

In the last section, we have investigated an interesting solution for the string moving on $\mathrm{AdS}_3\times \mathrm{S}^3$ with three angular momenta, one on the AdS space and two on the sphere. We have found two classes of solutions: the circular and the helical string solutions and find the relation among various conserved charges in a particular parameter space. In the special limit $v = 0$ and $\tilde{w} = 0$, the sphere part of the circular string on $\mathrm{AdS}_3 \times S^3$ reduces to the giant magnon on S^2, which is the half of the GKP folded string [41], with the infinite angular momentum S and the infinite winding numbers. Notice that circular and helical string solution do not satisfy the dispersion relation $E - S \sim \log S$, as in the case of GKP folded string. When $\tilde{w} \neq 0$, the circular string becomes one containing the finite angular momentum J_2 in the ϕ_2-direction. Moreover, the angular momentum J_2 is related to the string world sheet momentum p as shown in eq. (4.37). From the conserved charges, we obtained the exact dispersion relation for the circular and helical string with two parameters, v and $\tilde{w}$, which has been rewritten in terms of the angular momenta S and J.

For the helical string which is an array of the infinitely many spike solutions rotating on the ϕ-direction, it also satisfies the exact dispersion relation given in the circular string case. In additions, we also obtained the dispersion relation for one spike of the helical string. Interestingly, the dispersion relation of the AdS spike is similar to that of the giant magnon on the sphere in that it has the infinite energy E and infinite angular momentum S but the difference of these, $E - S$ is given by the finite angle difference in the ϕ-direction.

In this paper, we have restricted all the parameters to a special region to make the calculation simple, so these solutions are not connected with the GKP string smoothly. So it will be an interesting work to find more general solution connected with the GKP string, which may shed light on obtaining deeper understanding for the GKP string and the corresponding dual gauge theory.

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References

[1] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[2] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [hep-th/0306139].

[3] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[4] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[5] K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through quadratic constraints*, *Commun. Math. Phys.* **46** (1976) 207.

[6] H.J. de Vega and N. Sanchez, *Exact integrability of strings in D-dimensional de Sitter space-time*, *Phys. Rev.* **D 47** (1993) 3394.

[7] M. Grigoriev and A.A. Tseytlin, *Pohlmeyer reduction of $AdS_5 \times S^5$ superstring σ-model*, *Nucl. Phys.* **B 800** (2008) 450 [arXiv:0711.0155].

[8] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys.* **A 39** (2006) 13095 [hep-th/0604135].

[9] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys.* **A 39** (2006) 13119 [hep-th/0604175].

[10] H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [hep-th/0605155].

[11] N.P. Bobev and R.C. Rashkov, *Multispin giant magnons*, *Phys. Rev.* **D 74** (2006) 046011 [hep-th/0607018].

[12] H.-Y. Chen, N. Dorey and K. Okamura, *On the scattering of magnon boundstates*, *JHEP* **11** (2006) 035 [hep-th/0608047].

[13] S. Hirano, *Fat magnon*, *JHEP* **04** (2007) 010 [hep-th/0610027].

[14] J.M. Maldacena and I. Swanson, *Connecting giant magnons to the pp-wave: an interpolating limit of $AdS_5 \times S^5$*, *Phys. Rev.* **D 76** (2007) 026002 [hep-th/0612079].

[15] J. Kluson, R.R. Nayak and K.L. Panigrahi, *Giant magnon in $NS5$-brane background*, *JHEP* **04** (2007) 099 [hep-th/0703244].

[16] J. Kluson and K.L. Panigrahi, *D1-brane in beta-deformed background*, *JHEP* **11** (2007) 011 [arXiv:0710.0148].

[17] D.M. Hofman and J.M. Maldacena, *Reflecting magnons*, *JHEP* **11** (2007) 063 [arXiv:0708.2272].

[18] N. Dorey and K. Okamura, *Singularities of the magnon boundstate S-matrix*, *JHEP* **03** (2008) 037 [arXiv:0712.4068].

[19] M. Kruczenski, *Spiky strings and single trace operators in gauge theories*, *JHEP* **08** (2005) 014 [hep-th/0410226].

[20] R. Ishizeki and M. Kruczenski, *Single spike solutions for strings on S^2 and S^3*, *Phys. Rev.* **D 76** (2007) 126006 [arXiv:0705.2429].

[21] M. Kruczenski and A.A. Tseytlin, *Spiky strings, light-like Wilson loops and pp-wave anomaly*, arXiv:0802.2039.

[22] A. Jevicki and K. Jin, *Solitons and AdS string solutions*, arXiv:0804.0412.

[23] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, *Generating AdS string solutions*, *JHEP* **03** (2008) 032 [arXiv:0712.1193].

[24] R. Ishizeki, M. Kruczenski, M. Spradlin and A. Volovich, *Scattering of single spikes*, *JHEP* **02** (2008) 009 [arXiv:0710.2300].

[25] M. Kruczenski, J. Russo and A.A. Tseytlin, *Spiky strings and giant magnons on S^5*, *JHEP* **10** (2006) 002 [hep-th/0607044].

[26] K. Okamura and R. Suzuki, *A perspective on classical strings from complex Sine-Gordon solitons*, *Phys. Rev.* **D 75** (2007) 046001 [hep-th/0609026].

[27] H. Hayashi, K. Okamura, R. Suzuki and B. Vicedo, *Large winding sector of AdS/CFT*, *JHEP* **11** (2007) 033 [arXiv:0709.4033].

[28] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, hep-th/0311139.

[29] M. Kruczenski and A.A. Tseytlin, *Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory*, *JHEP* **09** (2004) 038 [hep-th/0406189].

[30] A.A. Tseytlin, *Semiclassical strings and AdS/CFT*, hep-th/0409296.

[31] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192].

[32] I.Y. Park, A. Tirziu and A.A. Tseytlin, *Semiclassical circular strings in AdS_5 and 'long' gauge field strength operators*, *Phys. Rev.* **D 71** (2005) 126008 [hep-th/0505130].

[33] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 731** (2005) 1 [hep-th/0507021].

[34] J.A. Minahan, A. Tirziu and A.A. Tseytlin, *Infinite spin limit of semiclassical string states*, *JHEP* **08** (2006) 049 [hep-th/0606145].

[35] W.-H. Huang, *Giant magnons under NS-NS and Melvin fields*, *JHEP* **12** (2006) 040 [hep-th/0607161].

[36] W.-H. Huang, *Semiclassical rotating strings in electric and magnetic fields deformed $AdS_5 \times S^5$ spacetimes*, *Phys. Rev.* **D 73** (2006) 026007 [hep-th/0512117].

[37] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *JHEP* **05** (2000) 048 [hep-th/0003037].

[38] W.-H. Huang, *Multi-spin string solutions in magnetic-flux deformed $AdS_n \times S^m$ spacetime*, *JHEP* **12** (2005) 013 [hep-th/0510136].

[39] N.P. Bobev and R.C. Rashkov, *Spiky strings, giant magnons and β-deformations*, *Phys. Rev.* **D 76** (2007) 046008 [arXiv:0706.0442].

[40] S. Ryang, *Three-spin giant magnons in $AdS_5 \times S^5$*, *JHEP* **12** (2006) 043 [hep-th/0610037].

[41] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].